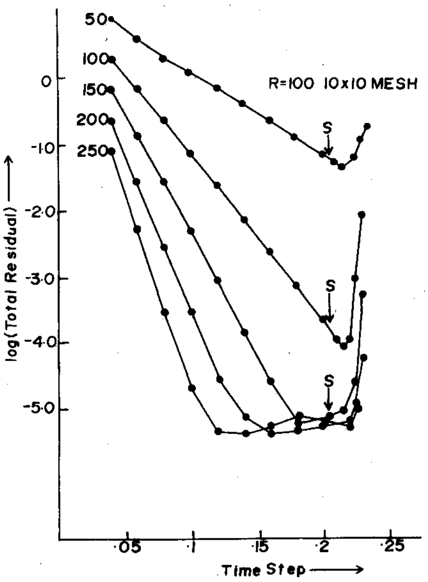

Fig. 1 Effect of step size on convergence. Each curve represents a fixed number of time steps and a constant amount of computer time. S = search prediction. Logarithms are to base ten.

References

[1] Richtmyer, R. D., and Morton, K. W., *Difference Methods for Initial-Value Problems*, 2nd ed., Interscience, New York, 1967, p. 16.

[2] Fromm, J. E., "A Method for Computing Nonsteady, Incompressible, Viscous Fluid Flows," LA-2910, Sept. 1963, Los Alamos Scientific Lab., Los Alamos, Calif.

[3] Zabransky, F., "Numerical Solutions for Cavity Flows," Ph.D. dissertation, 1968, Depart. of Applied Mathematics, Univ. of Western Ontario.

[4] Thom, A., and Apelt, C. J., *Field Computations in Engineering and Physics*, 1st ed., Van Nostrand, New York, 1961, pp. 123–137.

[5] Mills, R. D., "Numerical Solution of the Viscous Flow Equations for a Class of Closed Flows," *Journal the Royal Aeronautical Society*, Vol. 69, 1965, pp. 714–718.

[6] Pearson, C. E., "A Numerical Study of the Time-Dependent Viscous Flow Between Two Rotating Spheres," *Journal of Fluid Mechanics*, Vol. 28, 1967, pp. 323–336.

Similar Solutions in Vibrational Nonequilibrium Nozzle Flows

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Nomenclature

A = area ratio
C,D = constants, for a given gas
L = nozzle scale parameter ($L = r_*'/\tan\gamma$)
M = Mach number
m = molecular weight
p = pressure
R = gas constant
S_0 = reservoir entropy ($S_0 = S_0'/R$)
T_t = translational temperature
T_v = vibrational temperature
u = velocity ($u = u'/u_0'$)
u_0' = velocity defined as $(p_0'/\rho_0')^{1/2}$
x = distance along the nozzle axis ($x = x'/L$)
ρ = density
θ_v = characteristic vibrational temperature ($\theta_v = h\nu/k$)
ϕ = vibrational temperature function ($\phi = \theta_v/T_v'$)
ψ = translational temperature function ($\psi = \theta_v/T_t'$)

Subscripts

$*$ = nozzle throat
0 = reservoir

Superscript

$(\ )'$ = dimensional quantity

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Introduction

In the past decade a considerable amount of theoretical effort has been directed toward understanding the nonequilibrium flow effects in nozzles under steady flow conditions. The problem has been studied under the assumption of pseudo-one-dimensional, adiabatic, inviscid flow. In spite of these simplifying assumptions the solutions are far from being simple and are often plagued by many numerical procedural difficulties.[1–3] A comprehensive review of this problem is presented in Ref 4. More recently, a time-dependent analysis[5] has been proposed which circumvents some of the numerical difficulties but retains the problem of determining the flow quantities through a numerous-stepped process. The present state-of-the-art for solving vibrational nonequilibrium nozzle flows requires complex computer programs with which the flow variables are determined by numerical integration for any given initial and boundary conditions. However, this approach does not provide suitable theoretical comparisons for use by the experimentalist because of the many variables involved. Thus, it is apparent that general correlating parameters are needed.

In the present analysis, the governing equations, for a pseudo-one-dimensional nonequilibrium nozzle flow with vibrational energy relaxation but no dissociation, are transformed into a similar form by using a new independent variable η. It can be shown that the similar solutions, for a family of nozzle shapes and a specified gas, depend on two parameters, s_0 and λ, in addition to the independent variable η. However, the similar equations are further reduced to a near universal form by a transformation of the independent variable η to ξ so that the similar solutions depend on a single parameter χ with ξ as the independent variable. General similar solutions which can be used for all combinations of initial conditions are presented in a single graph for nitrogen.

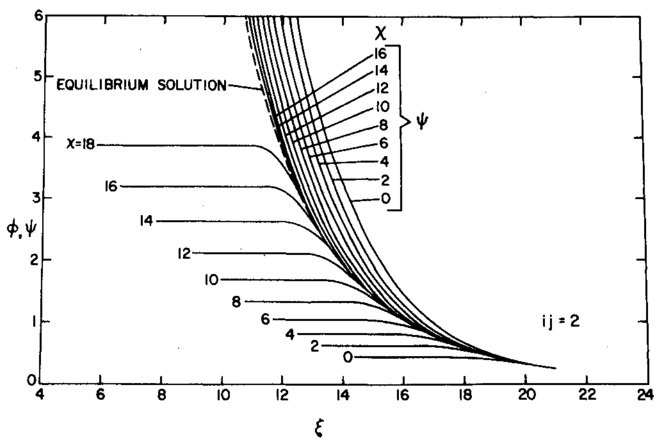

Fig. 1 Similar solutions for vibrational nonequilibrium nitrogen flow (D = 14.7).

The parameters χ and ξ are the exact and general correlating parameters.

Transformation of Governing Equations

The governing equations for a steady pseudo-one-dimensional, adiabatic flow with negligible dissociation are considered. The flow variables and nozzle physical characteristics are nondimensionalized with the corresponding reservoir values, the primed symbols denote dimensional quantities and the unprimed symbols represent nondimensional quantities. By using the Landau-Teller type vibrational relaxation rate equation with the relaxation rate time constant τ given as

$$\tau_v p' = Ce^{D(\theta v/Tt')^{1/3}} \quad (1)$$

the governing equations can be reduced to two differential equations,[6]

$$d\eta - (5/2)(d\psi/\psi) - (1 - f_1)\psi e^{\phi}d\phi/(e^{\phi} - 1)^2 = 0 \quad (2)$$

and the modified rate equation

$$\frac{d\phi}{d\eta} = \frac{1}{(N_s)_2} e^{\lambda_2}\psi^{-1} \exp - [\eta(1-1/ij)+\psi^{1/3}D+\phi] \times \left(\frac{e^{\phi}-1}{e^{\psi}-1}\right)(e^{\psi} - e^{\phi})\left(\frac{1-f_2}{1-f_1}\right) \quad (3)$$

where

$$\lambda_2 = \log_e[(\rho_* u_*)^{1/ij} L\rho_d u_d \theta/ijC]$$

and

$$(N_s)_2 = [M^2/(M^2 - 1)](1 - A^{-1/i})^{(j-1)/j} u^{(1+1/ij)}$$

In deriving Eqs. (2) and (3) the parameter η is defined as $\eta = \log_e(uA/\rho_* u_*)$ where $A = (1 + x^j)^i$. Further, the main motivation in expressing the rate equation in the form shown in Eq. (3) is to combine all the parameters of the problem into a single parameter λ_2. However, Eq. (3) also contains the additional parameters D and ij where D is a constant for a given gas and ij is the nozzle shape parameter ($ij = 2.0$ for nozzle shapes of practical interest). The nonsimilar function, $(N_s)_2$ given in Eq. (3), is a function of M, u and A. Hence, it will have different values for different reservoir conditions and it also varies along the nozzle axis. The properties of $(N_s)_2$ and the details of a method used for including the effects of $(N_s)_2$ in the solutions of the similar Eqs. (2) and (3) are presented in Ref. 6. Essentially, it is shown that the expression for $(N_s)_2$ can be modified and expressed in terms of a new parameter ξ, where $\xi = (S_0 - \eta)$, and given as

$$N_s = 0.37 - 0.32(2.0 - \xi_*/\xi)^{6.6}$$

for

$$1 < (\xi_*/\xi) < 2.0 \quad (4)$$

and

$$N_s = 0.37 \text{ for } (\xi_*/\xi) > 2.0$$

The f functions appearing in Eqs. (2) and (3) take into account the effect of the cutoff harmonic oscillator approximation which is assumed. It can be shown that for temperatures as high as 7000°K for nitrogen, these f functions are negligible.[6]

The problem under consideration has been reduced to solving two similar differential equations, Eqs. (2) and (3), for two unknowns ψ and ϕ with η as the independent variable. Once ψ and ϕ are determined by solving Eqs. (2) and (3) the other unknowns p, ρ and u are obtained from the other governing equations which are simple algebraic expressions.

Equilibrium Flow Solution

The equilibrium flow solution is a limiting case which is achieved when the vibrational relaxation time τ_v is very short

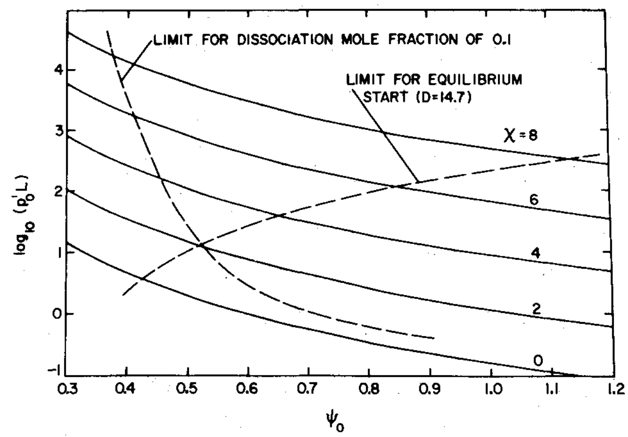

Fig. 2 Range of applicability of the parameter χ ($ij = 2.0$, $C = 1.715 \times 10^{-11}$ atm-sec, p_0' atm, L cm).

(when $\lambda_2 \to \infty$). For this situation, $\phi = \psi$ and the generalized momentum equation [Eq. (2)] can be integrated and given as

$$\eta - (5/2) \log_e\psi - \log_e(1 - e^{-\psi}) + \psi/(e^{\psi} - 1) + f_3 = \text{const} \quad (5)$$

where f_3 is a term associated with the cutoff harmonic oscillator approximation. The expression for the corresponding entropy can be shown as

$$S_0 \equiv (S_0'/R) = \eta - (5/2) \log_e\psi - \log_e(1 - e^{-\psi}) + \psi/(e^{\psi} - 1) + f_3 + S_r \quad (6)$$

Where $S_r = S_r'/R$ = reference entropy. The equilibrium flow solution can be obtained from Eq. (5) with η as the independent variable and S_0 as the parameter. It is noted that this solution does not depend on the nozzle geometry.

Nonequilibrium Flow Solution

The initial values needed for solving Eqs. (2) and (3) are obtained in the following manner. The flow is assumed to be in an equilibrium condition up to the throat. In this case the function $\phi = \psi$ and the initial value of ψ can be obtained from the equilibrium solution [Eq. (5)] for given values of η and S_0. In this rather indirect way, the entropy S_0 also appears as a parameter in the nonequilibrium solutions.

For a given gas (D constant) and family of nozzle shapes (ij constant) the parameters λ_2 and S_0 have to be specified for the nonequilibrium solutions. However, this two parametric dependence can be reduced to a single one with the introduction of a new transformation variable ξ defined as $\xi = (S_0 - \eta)$. The Eq. (5), which gives the equilibrium solutions, reduces to

$$\xi = \psi/(e^{\psi} - 1) - \log_e[\psi^{5/2}(1 - e^{-\psi})] + f_3 + S_r \quad (7)$$

Thus, the equilibrium solutions may be represented by a single general curve showing the variation of ψ with the independent variable ξ. The governing equations [Eqs. (2) and (3)] for the nonequilibrium case reduce to

$$(5/2)d(\log_e\psi) + d\xi + \psi[e^{\phi}/(e^{\phi} - 1)^2]d\phi(1 - f_1) = 0 \quad (8)$$

and

$$d\phi/d\xi = [1/(N_s)_2]e^{\chi_2}\psi^{-1} \exp^{[(1-1/ij)\xi - D\psi^{1/3} - \phi]} \times [(e^{\phi} - 1)/(e^{\psi} - 1)](e^{\phi} - e^{\psi})[(1 - f_2)/(1 - f_1)] \quad (9)$$

where

$$\chi_2 = [\lambda_2 - (1 - 1/ij)S_0]$$

Discussion of Nonequilibrium Solutions

The two similar governing equations [Eqs. (8) and (9)] for the nonequilibrium flow case, with N_s given by Eq. (4), were solved by a fourth-order Runge-Kutta technique with $ij = 2.0$ and $D = 14.7$ which corresponds to nitrogen. The starting values of ψ and ϕ were obtained for a given ξ from the equilibrium solution ($\psi = \phi$) given by Eq. (7). The reference entropy S_r value in Eq. (7) was taken as 15.5. The factors f_1, f_2 and f_3 were all assumed to be zero. The starting values of ξ for different χ values were selected in such a way that the solution always started with equilibrium conditions.

A series of solutions for different values of χ for nitrogen are shown in Fig. 1. The vibrational temperature function ϕ is seen to follow the translational function ψ very closely for awhile, the extent of which depends on χ and then diverges rather suddenly and reaches a constant value, this corresponds to the freezing of the vibrational energy mode. The translational temperature function ψ increases monotonically as ξ decreases. The equilibrium solution shown in Fig 1 is also given by the envelope of all the nonequilibrium solutions.

It has been shown that the nonequilibrium similar solutions depend on two general parameters ξ and χ. In order to use the similar solutions presented in this paper (Fig. 1) the parameters ξ and χ should be known in terms of the initial and boundary values. Therefore, the functional dependence of ξ and χ must be considered. The parameter ξ is a function of not only the reservoir and nozzle throat conditions, but also of velocity. In Ref. 6 it is shown that a modified velocity ratio u'/u_*' can be well correlated with the area ratio A'/A_*'. This leads to a simple expression for a modified velocity ratio which is used in finally expressing ξ as

$$\xi = \psi_0/(e^{\psi_0} - 1) - \log_e\{k_1^{-4}k_2^{-1}\psi_0^{5/2}(1 - e^{-\psi_0}) \times A[0.5 - 0.31(1 + \log_{10}A)^{-2}]\} + S_r \quad (10)$$

(The constants k_1 and k_2 are given in Ref. 6.) It is noted that ξ depends on only two parameters, namely, the area ratio and the reservoir temperature function ψ_0. The parameter ξ is now expressed in terms of the initial and boundary values only and hence can be readily computed for any given conditions.

The parameter χ can be shown[6] to reduce to

$$\chi = \log_e\left[k_1^{(6+1/ij)}\left(\frac{m}{R\theta_v}\right)^{1/2}\left(\frac{p_0'}{C}\right)L\psi_0^{(4-2.5/ij)} \times (1 - e^{-\psi_0})^{(1-1/ij)}\right] - \left(1 - \frac{1}{ij}\right)\left[\frac{\psi_0}{e^{\psi_0} - 1} + S_r\right] \quad (11)$$

It is noted that χ is independent of A. Also, for a given gas χ depends on only p_0', L and ψ_0 since θ_0, C and S_r are all constants.

To apply the results presented here, ξ and χ values can be computed using Eqs. (10) and (11) based on known reservoir conditions and area ratio. Then, using these ξ and χ values, along with Fig. 1, ϕ and ψ values can be determined with which the algebraic governing equations can be used to determine p, ρ and u.

Range of Applicability of the Parameter χ

The general correlating parameter χ depends on p_0', L and ψ_0 for a given gas. The variation of $p_0'L$ with ψ for a constant χ was computed using Eq. (11) and is shown in Fig. 2 for a number of χ values. The variation of $p_0'(L = 1.0)$ with ψ_0 for a constant equilibrium mole fraction of 0.1 is also shown; this curve represents approximately the high temperature limit beyond which dissociation relaxation may have to be considered. A curve is also shown which indicates the maximum ψ value, for a given χ, where the nonequilibrium solution departs from the equilibrium solution. For reservoir conditions defined by the region above this line, the solutions start with equilibrium conditions and can be obtained from the present similar solutions. For reservoir conditions defined by the region well below this line the flow can be considered as frozen in the entire nozzle. In a narrow region just below the equilibrium limit line the flow will be in the nonequilibrium state and the solutions have to be obtained by starting with the reservoir conditions as the initial values.

Conclusions

Based on the present analysis the following conclusions are reached. 1) Similar solutions for vibrational nonequilibrium nozzle problems can be obtained over a wide range of initial conditions and nozzle scale parameters by using the new similarity parameter ξ. 2) The parameters ξ and χ serve as general correlating parameters since they contain all the parameters of the problem. The vibrational equilibrium solutions depend on the one parameter χ only and the nonequilibrium solutions depend on two parameters χ and ξ.

References

[1] Stollery, J. L. and Smith, J. E., "A Note on the Vibration of Vibrational Temperature Along a Nozzle," *Journal of Fluid Mechanics*, Vol. 13, 1962, pp. 225–235.

[2] Stollery, J. L. and Park, C., "Computer Solutions to the Problem of Vibrational Relaxation in Hypersonic Nozzle Flows," *Journal of Fluid Mechanics*, Vol. 19, 1964, pp. 113–123.

[3] Erickson, W. D., "Vibrational-Nonequilibrium-Flow of Nitrogen in Hypersonic Nozzles," TN D-1810, June 1963, NASA.

[4] Hall, J. G. and Treanor, C. E., "Nonequilibrium Effects in Supersonic-Nozzle Flows," Rept. 163, March 1968, Cornell Aeronautical Lab., Buffalo, N.Y.

[5] Anderson, J. D., "A Time-Dependent Analysis for Vibrational and Chemical Nonequilibrium Nozzle Flows," *AIAA Journal*, Vol. 8, No. 3, March 1970, pp. 545–550.

[6] Reddy, N. M. and Daum, F. L., "Similar Solutions in Vibrational Nonequilibrium Nozzle Flows," AIAA Paper 70-804, Los Angeles, Calif., 1970.

Time-Dependent Numerical Analysis of MHD Blunt Body Problem

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THE interaction of shock heated plasma and an onboard magnet may prove attractive for an atmospheric entry vehicle (Fig. 1). A review of the subject is contained in Ref. 1.

In the present work, the time dependent finite difference method[2] is applied. Results are compared with those of

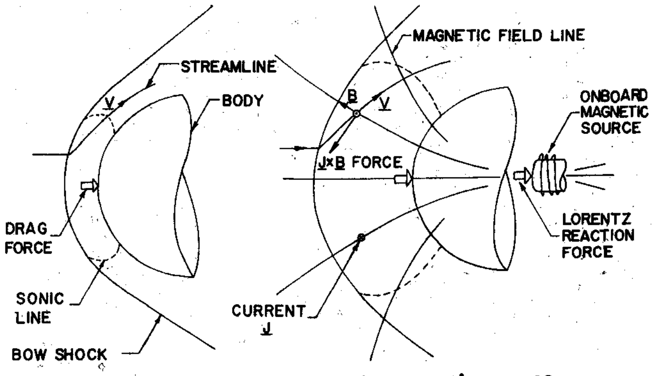

Fig. 1 Effect of onboard magnetic source.

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